

# Asymptotic limits for mildly degenerate Kirchhoff equations

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## Abstract

We consider the second order Cauchy problem

$$\varepsilon u''_\varepsilon + |A^{1/2}u_\varepsilon|^{2\gamma}Au_\varepsilon + u'_\varepsilon = 0, \quad u_\varepsilon(0) = u_0 \neq 0, \quad u'_\varepsilon(0) = u_1$$

where  $\varepsilon > 0$ ,  $H$  is an Hilbert space,  $A$  is a self-adjoint positive operator on  $H$  with dense domain  $D(A)$ ,  $(u_0, u_1) \in D(A) \times D(A^{1/2})$ , and  $\gamma > 0$ .

We study accurately the decay as  $t$  goes to infinity of the solutions, provided that  $\varepsilon$  is small enough. In particular we obtain a new estimate on  $u''_\varepsilon$  and we show that the renormalized functions  $(1+t)^{1/(2\gamma)}(u_\varepsilon(t), (1+t)u'_\varepsilon(t))$  have a non zero limit in  $D(A) \times D(A^{1/2})$  as  $t$  goes to infinity. Moreover we calculate explicitly the norm of these limit and we prove that they do not depend on the initial data.

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**Key words:** degenerate damped hyperbolic equations, Kirchhoff equations, decay rate of solutions.

# 1 Introduction

Let  $H$  be a real Hilbert space. Given  $x$  and  $y$  in  $H$ ,  $|x|$  denotes the norm of  $x$ , and  $\langle x, y \rangle$  denotes the scalar product of  $x$  and  $y$ . Let  $A$  be a self-adjoint linear operator on  $H$  with dense domain  $D(A)$ . We always assume that  $A$  is coercive, namely  $\langle Au, u \rangle \geq \sigma_0 |u|^2$  for every  $u \in D(A)$ . For any such operator the power  $A^\alpha$  is defined for every  $\alpha \geq 0$  in a suitable domain  $D(A^\alpha)$ .

For every  $\varepsilon > 0$  we consider the second order Cauchy problem

$$\varepsilon u_\varepsilon''(t) + |A^{1/2}u_\varepsilon(t)|^{2\gamma} Au_\varepsilon(t) + u_\varepsilon'(t) = 0, \quad \forall t \geq 0, \quad (1.1)$$

$$u_\varepsilon(0) = u_0 \neq 0, \quad u_\varepsilon'(0) = u_1, \quad (1.2)$$

where  $u_0 \in D(A)$ ,  $u_1 \in D(A^{1/2})$ . This problem is just an abstract setting (with  $m(r) = r^\gamma$ ) of the initial boundary value problem for the hyperbolic partial differential equation (PDE)

$$\varepsilon u_{tt}^\varepsilon(t, x) - m\left(\int_\Omega |\nabla u^\varepsilon(t, x)|^2 dx\right) \Delta u^\varepsilon(t, x) + u_t^\varepsilon(t, x) = 0 \quad (1.3)$$

in a bounded open set  $\Omega \subseteq \mathbb{R}^n$ , where  $m$  is a non negative function. This equation is a model for the damped small transversal vibrations of an elastic string ( $n = 1$ ) or membrane ( $n = 2$ ) with uniform density  $\varepsilon$ .

Equations as (1.1) or (1.3) have been intensely studied from '80 both in the case of an operator  $A$  coercive and in the case of an only nonnegative operator. In particular it was extensively considered the case of a function  $m$  in the  $C^1$  class, both in the nondegenerate case ( $m \geq c > 0$ ) and in the mildly degenerate case ( $m(|A^{1/2}u_0|^2) \neq 0$ ). For a more complete discussion on this argument we refer to the survey [8] and to the references contained therein. Here we concentrate shortly only on few of the results concerning the existence of global solutions and their behaviour at the infinity. First of all let us remind that there are not substantial differences between coercive and only nonnegative operators with respect to the existence of global solutions, while there are differences regarding the asymptotic behaviour. Indeed in the case of only nonnegative operators the estimates that one can obtain on  $|A^{1/2}u_\varepsilon|$  are in general worse (see [6]). All the results we state explicitly regarding the decay of the solutions must then be thought in the coercive case. We point out that the existence of a global solution for small data was proved firstly in the nondegenerate case of  $m \geq c > 0$  in [1] and in [18], then in the mildly degenerate case of (1.1) - (1.2) by K. NISHIHARA AND Y. YAMADA [14] if  $\gamma \geq 1$  and in [3] - [4] when  $0 < \gamma < 1$ . In all these papers it was also considered the behaviour at the infinity of the solutions and some, in general non optimal, estimates were obtained (see also [2], [9], [10] [13] and [17] for the nondegenerate case). In the mildly degenerate case of (1.1) good estimates on  $|A^{1/2}u_\varepsilon|$ ,  $|Au_\varepsilon|$ ,  $|u_\varepsilon'|$  were proved firstly by T. MIZUMACHI ([11], [12]) and K. ONO ([15], [16]) when  $\gamma = 1$  and then for any  $\gamma > 0$  in [6], [7] (see Theorem 1 for the precise statement in the coercive case). It is clear

that estimates on  $|A^{1/2}u_\varepsilon|$ ,  $|Au_\varepsilon|$ ,  $|u'_\varepsilon|$  produce estimates also on  $u''_\varepsilon$ . These estimates are in general not sharp, as it was shown when  $\gamma = 1$  in [15]. Indeed in this last case the obvious estimate gives  $(1+t)^3|u''_\varepsilon(t)|^2 \leq C_\varepsilon$ , while K. Ono proved that at least one has  $(1+t)^4|u''_\varepsilon(t)|^2 \leq C_\varepsilon$ .

We proposed to clarify more precisely the behaviour of the solutions of (1.1) - (1.2), by studying the problem from a new point of view. We prove indeed that not only estimates as (2.1)-(2.3) hold true, but in fact the renormalized functions  $(1+t)^{1/(2\gamma)}(u_\varepsilon(t), (1+t)u'_\varepsilon(t))$  have a *non zero* limit  $(u_{\varepsilon,\infty}, v_{\varepsilon,\infty})$  in  $D(A) \times D(A^{1/2})$ . Of such limits we can calculate explicitly the norms, that do not depend on the initial data or  $\varepsilon$ . As a consequence we obtain sharp estimates on the decay of the solutions of (1.1) in  $D(A) \times D(A^{1/2})$ . Equally important we can express explicitly the relation between  $u_{\varepsilon,\infty}$  and  $v_{\varepsilon,\infty}$ . This allows us also to obtain better estimates on  $u''_\varepsilon$ , indeed for example in the case of  $\gamma = 1$  our estimates give (see Theorem 2.3):

$$(1+t)^5|u''_\varepsilon(t)|^2 \leq C_\varepsilon.$$

For our purpose in the following we assume that  $H$  has a countable basis made by eigenvectors of  $A$ , that is obviously verified in the concrete case of (1.3). It is well known that in fact it is enough to assume that initial data in (1.2) can be written in Fourier series with respect to eigenvectors of  $A$ , because in such a case this is also true for the solution of (1.1). This hypothesis allows us to prove (see Theorem 2.3 and (2.19), (2.20)) that actually the norms of  $u_{\varepsilon,\infty}$  in  $D(A)$  and of  $v_{\varepsilon,\infty}$  in  $D(A^{1/2})$  depend only on  $\gamma$  and on the smallest eigenvalue of  $A$ . This surprising behaviour depends on the fact that actually the behaviour of the solutions of (1.1) is due only to the components of  $u_\varepsilon$  related to the smallest eigenvalue of  $A$  (see Theorem 2.1). This type of estimates so precise should allow us to obtain decay-error estimates in the study of the singular perturbation problem, that consists in setting formally  $\varepsilon = 0$  in (1.1) and study the difference between  $u_\varepsilon$  and the solution of the new first order problem obtained in such a way (see [8] and [9] for an introduction of the problem and its treatment in the non degenerate case).

The outline of the paper is the following. In Section 2 to begin with we recall the hold result we need on the existence of global solutions of (1.1) and their decay and we introduce some preliminary notations, then we state the main results. In section 3 we prove the results. This last section is divided in various parts. First of all we prove a general linear result that we use in particular for proving Theorem 2.1, then we study the properties of the components of the solution of (1.1) and finally prove Theorem 2.3. Let us stress that Theorem 2.1 is in fact a linear result whereas proof of Theorem 2.3 requires a new type of nonlinear approach.

## 2 Statements

### 2.1 Notations and preliminaries

Let us stress that we assumed that the operator  $A$  is coercive. The following result is well known and it is a consequence of [14], [4] (see also [5] for the study of the case of more general functions  $m$ ) for the part concerning the existence of global solutions, while it follows from [6] [7] (see also [11], [12], [14], [15], [16]) for the part concerning the decay of solutions.

**Theorem 1** Let  $(u_0, u_1) \in D(A) \times D(A^{1/2})$ . Let  $\gamma > 0$ , then for  $\varepsilon$  small the mildly degenerate problem (1.1), (1.2) has a unique global solution

$$u_\varepsilon \in C^2([0, +\infty[, H) \cap C^1([0, +\infty[, D(A^{1/2})) \cap C^0([0, +\infty[, D(A))$$

such that

$$\frac{K_1}{(1+t)^{1/\gamma}} \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{1/\gamma}} \quad \forall t \geq 0, \quad (2.1)$$

$$\frac{K_1}{(1+t)^{1/\gamma}} \leq |Au_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{1/\gamma}} \quad \forall t \geq 0, \quad (2.2)$$

$$|u'_\varepsilon(t)|^2 \leq \frac{K_2}{(1+t)^{2+1/\gamma}} \quad \forall t \geq 0; \quad (2.3)$$

where the constants  $K_1$  and  $K_2$  do not depend on  $\varepsilon$ .

In the following we assume always that  $\varepsilon \leq 1$  is small enough so that Theorem 1 holds true.

Let us now set

$$b_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^{2\gamma}, \quad b_\varepsilon(0) = |A^{1/2}u_0|^{2\gamma} =: b_0. \quad (2.4)$$

An immediate consequence of Theorem 1 is that

$$\frac{K_3}{1+t} \leq b_\varepsilon(t) \leq \frac{K_4}{1+t}, \quad \frac{|b'_\varepsilon(t)|}{b_\varepsilon(t)} \leq \frac{K_4}{1+t} \quad \forall t \geq 0, \quad (2.5)$$

where the constants  $K_3$  and  $K_4$  do not depend on  $\varepsilon$ .

Moreover let us define

$$B_\varepsilon(t) = \int_0^t b_\varepsilon(s) ds. \quad (2.6)$$

From (2.5) we know that  $B_\varepsilon(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , and we have also a non optimal but  $\varepsilon$  independent estimate on the speed wherewith it diverges. We propose to obtain sharp estimates on the behaviour of  $B_\varepsilon$  and, as a consequence, also on  $u_\varepsilon$ .

Before proceeding, we introduce some general notations. Let  $(e_k)_k$  be a countable basis of  $H$  made by eigenvectors of  $A$ , and  $\lambda_k^2$  be the corresponding eigenvalues, that is

$$Ae_k = \lambda_k^2 e_k, \quad \forall k;$$

therefore for every  $u \in H$  we have that

$$u = \sum_k u_k e_k.$$

In particular if  $u_\varepsilon$  is the solution of (1.1), (1.2), then

$$u_\varepsilon(t) = \sum_k u_{\varepsilon,k}(t) e_k,$$

where  $u_{\varepsilon,k}$  solves

$$\varepsilon u_{\varepsilon,k}''(t) + b_\varepsilon(t) \lambda_k^2 u_{\varepsilon,k}(t) + u_{\varepsilon,k}'(t) = 0, \quad u_{\varepsilon,k}(0) = u_{0,k}, \quad u_{\varepsilon,k}'(0) = u_{1,k}. \quad (2.7)$$

Let us now define for  $\lambda > 0$ :

$$H_\lambda := \left\{ u \in H : u = \sum_{k: \lambda_k \geq \lambda} u_k e_k \right\}, \quad H_{\{\lambda\}} := \left\{ u \in H : u = \sum_{k: \lambda_k = \lambda} u_k e_k \right\},$$

$$H_{[\lambda, \mu)} := \left\{ u \in H : u = \sum_{k: \lambda \leq \lambda_k < \mu} u_k e_k \right\},$$

and

$$A_\lambda = A|_{H_\lambda}, \quad A_{\{\lambda\}} = A|_{H_{\{\lambda\}}}, \quad A_{[\lambda, \mu)} = A|_{H_{[\lambda, \mu)}}.$$

Let moreover  $\nu$  be defined by:

$$\nu := \min\{\lambda_k : u_{0,k} \neq 0, \text{ or } u_{1,k} \neq 0\}.$$

Since the components of  $u_\varepsilon$  solves (2.7) then we can assume sine loss of generality that  $\nu^2$  is the smallest eigenvalue of  $A$ , that is  $A = A_\nu$ , and that

$$\langle Au, u \rangle \geq \nu^2 |u|^2 \quad \forall u \in D(A). \quad (2.8)$$

Moreover for every  $\mu > \nu$  we can decompose  $u \in H$  as

$$u = u_\nu + \bar{u}_\mu + U_\mu \quad (2.9)$$

where  $u_\nu \in H_{\{\nu\}}$ , and  $U_\mu \in H_\mu$ .

Finally for every  $\lambda \geq \nu$  let us define the corrector  $\Theta_{\varepsilon, \lambda} \in H_\lambda$  as the solution of

$$\varepsilon \Theta_{\varepsilon, \lambda}'' + \Theta_{\varepsilon, \lambda}' = 0, \quad \Theta_{\varepsilon, \lambda}(0) = 0, \quad \Theta_{\varepsilon, \lambda}'(0) = U_{1, \lambda} + b_0 A_\lambda U_{0, \lambda}. \quad (2.10)$$

## 2.2 Statements

We are now ready to state our results. The first one concerns the decay of the components of  $u_\varepsilon$ .

**Theorem 2.1** *Let  $u_\varepsilon$  be the solution of (1.1), (1.2) as in Theorem 1 with  $(u_0, u_1) \in D(A) \times D(A^{1/2})$  and let  $\lambda \geq \nu$ . Then for  $\varepsilon$  small (depending on  $\lambda$ ) we have the following inequalities.*

1. For  $h = 0, 1$  there exist constants  $\gamma_{h,\lambda}$  independent of  $\varepsilon$  such that:

$$e^{2\lambda^2 B_\varepsilon(t)} \left( \varepsilon \frac{|A^{h/2} U'_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon(t)} + |A^{(h+1)/2} U_{\varepsilon,\lambda}(t)|^2 \right) \leq \gamma_{h,\lambda} \quad \forall t \geq 0. \quad (2.11)$$

2. There exists a constant  $\gamma_\lambda$  independent of  $\varepsilon$  such that:

$$e^{2\lambda^2 B_\varepsilon(t)} \frac{|U'_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon^2(t)} \leq \gamma_\lambda \quad \forall t \geq 0. \quad (2.12)$$

3. There exists a constant  $\gamma_{\varepsilon,\lambda}$  such that:

$$e^{2\lambda^2 B_\varepsilon(t)} \frac{|U''_{\varepsilon,\lambda}(t) - \Theta''_{\varepsilon,\lambda}(t)|^2}{b_\varepsilon^4(t)} \leq \gamma_{\varepsilon,\lambda} \quad \forall t \geq 0. \quad (2.13)$$

Moreover if  $(u_0, u_1) \in D(A^2) \times D(A^{3/2})$ , then we can take also  $\gamma_{\varepsilon,\lambda}$  independent of  $\varepsilon$ .

**Remark 2.2** Theorem 2.1 is in fact a linear result, indeed in the proof we use only that  $u_{\varepsilon,k}$  verifies (2.7) for every  $k$  with a coefficient  $b_\varepsilon$  that satisfies (2.5). This means that if the initial data are more regular then estimates like (2.11) - (2.12) - (2.13) hold true also for  $A^{h/2} U_{\varepsilon,\lambda}$  with suitable large  $h$ .

Theorem 2.1 says that the components of  $u_\varepsilon$  related to big eigenvalues decay faster of the component related to the smallest one and as more faster it depends on the behaviour of  $B_\varepsilon$ . The following result clarifies this aspect.

**Theorem 2.3** *Let  $u_\varepsilon$  be the solution of (1.1), (1.2) as in Theorem 1 with  $(u_0, u_1) \in D(A) \times D(A^{1/2})$ . Then for  $\varepsilon$  small there exists a non zero vector  $u_{\varepsilon,\infty} \in H_{\{\nu\}}$  such that as  $t \rightarrow +\infty$ :*

$$(1+t)^{1/(2\gamma)} (u_\varepsilon(t), (1+t)u'_\varepsilon(t)) \rightarrow (u_{\varepsilon,\infty}, -\frac{1}{2\gamma}u_{\varepsilon,\infty}), \quad \text{in } D(A) \times D(A^{1/2}). \quad (2.14)$$

Moreover the following properties hold true.

1. There exist constants  $K_{\varepsilon,1}$ ,  $K_{\varepsilon,2}$  such that for all  $t \geq 0$  we have:

$$\frac{K_{\varepsilon,1}}{1+t} \leq e^{-2\nu^2\gamma B_{\varepsilon}(t)} \leq \frac{K_{\varepsilon,2}}{1+t}. \quad (2.15)$$

Furthermore if  $u_{0,\nu} \neq 0$  then we can take  $K_{\varepsilon,1}$  and  $K_{\varepsilon,2}$  independent of  $\varepsilon$ .

2. The following limits hold true for  $t \rightarrow +\infty$ :

$$(1+t)b_{\varepsilon}(t) \rightarrow \frac{1}{2\nu^2\gamma}; \quad (2.16)$$

$$(1+t)^{1/\gamma}|u_{\varepsilon,\nu}(t)|^2 \rightarrow \frac{1}{\nu^2(2\nu^2\gamma)^{1/\gamma}}. \quad (2.17)$$

3. There exists a constant  $K_{\varepsilon}$  such that for all  $t \geq 0$  we have

$$|u_{\varepsilon}''(t)|^2 \leq K_{\varepsilon} \frac{1}{(1+t)^{4+1/\gamma}} \quad (2.18)$$

Let us now give some observations on Theorem 2.3.

- Inequalities (2.15) together with (2.17) and Theorem 2.1 say that how much  $U_{\varepsilon,\lambda}$  decay faster of  $u_{\varepsilon,\nu}$  depends on  $\lambda^2/\nu^2$ .
- From limits in (2.14) and (2.17) we have that

$$|u_{\varepsilon,\infty}|^2 = \frac{1}{\nu^2(2\nu^2\gamma)^{1/\gamma}},$$

and it is also obvious that as  $t \rightarrow +\infty$  we have that

$$(1+t)^{1/\gamma}|A^{1/2}u_{\varepsilon}(t)|^2 \rightarrow \frac{1}{(2\nu^2\gamma)^{1/\gamma}}; \quad (1+t)^{1/\gamma}|Au_{\varepsilon}(t)|^2 \rightarrow \frac{\nu^2}{(2\nu^2\gamma)^{1/\gamma}}; \quad (2.19)$$

$$(1+t)^{2+1/\gamma}|u_{\varepsilon}'(t)|^2 \rightarrow \frac{\nu^2}{(2\nu^2\gamma)^{2+1/\gamma}}, \quad (1+t)^{2+1/\gamma}|A^{1/2}u_{\varepsilon}'(t)|^2 \rightarrow \frac{\nu^4}{(2\nu^2\gamma)^{2+1/\gamma}}, \quad (2.20)$$

hence the behaviour at the infinity of the norms do not depends on the initial conditions.

- Limits in (2.19) clarify the estimates in (2.1) and (2.2) and show that the estimate in (2.3) is sharp and that a similar estimate holds true (maybe with constants depending on  $\varepsilon$ ) also for  $|A^{1/2}u_{\varepsilon}'|$ .



- The estimate in (2.18) looks better of all the previous known on the second derivative of  $u_\varepsilon$ , moreover it seems optimal, indeed the rate decay is the same as in the limit case  $\varepsilon = 0$  with only one real component (in such a case we have only to solve the ordinary real differential equation  $y' + \nu^2 y^{2\gamma+1} = 0$ ).
- Theorem 2.1 and Theorem 2.3 say that as in the case of a linear equation (with a coefficient  $b(t)$  verifying (2.5)) the decay of the solution is decided only by the smallest eigenvalue  $\nu^2$  for which there are non zero components of the initial data. Nevertheless in our case the decay rate does not depend on  $\nu^2$ .

### 3 Proofs

In some of the proofs we employ the following simple comparison result that has already been used in various forms in a lot of papers, starting from [5].

**Lemma 3.1** *Let  $f \in C^1([0, +\infty))$  and let us assume that  $f(t) \geq 0$  in  $[0, +\infty)$ , and that there exist two constants  $K_5 > 0$ ,  $K_6 \geq 0$  such that*

$$f'(t) \leq -K_5 \sqrt{f(t)} \left( \sqrt{f(t)} - K_6 \right) \quad \forall t \geq 0.$$

*Then we have that  $f(t) \leq \max \{f(0), K_6^2\}$  for every  $t \geq 0$ .*

We divide the proofs in various parts. First we prove two basic propositions on linear equations. Then we prove Theorem 2.1. After we study the decomposition of  $u_\varepsilon$  made by (2.9) and finally we prove Theorem 2.3.

#### 3.1 Linear equations and estimates

Let  $M$  be a self-adjoint linear operator on  $H$ . Let us assume that

$$\langle Mw, w \rangle \geq \sigma_M^2 |w|^2 \quad \forall w \in D(M). \quad (3.1)$$

For  $h \geq 0$  let us denote by  $|w|_{D(M^h)}$  the norm of the vector  $w$  in the space  $D(M^h)$ .

Let us assume that  $b : [0, +\infty[ \rightarrow ]0, +\infty[$  is a  $C^1$  function that verifies

$$b(t) \leq \frac{K_4}{1+t}, \quad \frac{|b'(t)|}{b(t)} \leq \frac{K_4}{1+t}, \quad \frac{|b'(t)|}{b^2(t)} \leq \frac{K_4}{K_3} \quad \forall t \geq 0, \quad (3.2)$$

where  $K_4$  and  $K_3$  are the constants in (2.5). For simplicity in the following we use these notations:

$$\|u\|^2 = |u|^2 (1 + b(0)^{-1} + b(0)^{-2}) \quad \text{if } u \in H,$$

$$\|u\|_{D(M^{h/2})}^2 = |u|_{D(M^{h/2})}^2 (1 + b(0)^{-1} + b(0)^{-2}) \quad \text{if } u \in D(M^{h/2}).$$

Let  $v_\varepsilon \in C^2([0, +\infty[, H) \cap C^1([0, +\infty[, D(M^{1/2})) \cap C^0([0, +\infty[, D(M))$  be the solution of the problem:

$$\varepsilon v_\varepsilon''(t) + b(t)Mv_\varepsilon(t) + v_\varepsilon'(t) = 0, \quad v_\varepsilon(0) = v_0 \in D(M), \quad v_\varepsilon'(0) = v_1 \in D(M^{1/2}). \quad (3.3)$$

Moreover let  $\theta_\varepsilon$  be the solution of

$$\varepsilon \theta_\varepsilon''(t) + \theta_\varepsilon'(t) = 0, \quad \theta_\varepsilon(0) = 0, \quad \theta_\varepsilon'(0) = v_1 + b(0)Mv_0, \quad (3.4)$$

so that  $\theta_\varepsilon(t) = \varepsilon \theta_\varepsilon'(0)(1 - e^{-t/\varepsilon})$ , and let us set

$$w_\varepsilon = v_\varepsilon - \theta_\varepsilon.$$

Finally let  $B$  defined as in (2.6) (using  $b(t)$  in place of  $b_\varepsilon$  of course).

Therefore the following propositions hold true.

**Proposition 3.2** *Let  $h \geq 1$  and let us assume that  $(v_0, v_1) \in D(M^{(h+1)/2}) \times D(M^{h/2})$ . Then for  $\varepsilon$  small depending only on  $\sigma_M^2$ ,  $K_3$  and  $K_4$  (and not on the initial data or  $h$ ), for all  $t \geq 0$  we have that:*

$$e^{2\sigma_M^2 B(t)} \left( \varepsilon \frac{|M^{h/2}v_\varepsilon'(t)|^2}{b(t)} + |M^{(h+1)/2}v_\varepsilon(t)|^2 \right) \leq L_0(\|v_1\|_{D(M^{h/2})}^2 + |v_0|_{D(M^{(h+1)/2})}^2) =: L_{h,M}, \quad (3.5)$$

$$e^{2\sigma_M^2 B(t)} \frac{|M^{(h-1)/2}v_\varepsilon'(t)|^2}{b^2(t)} \leq L_1(\|v_1\|_{D(M^{h/2})}^2 + |v_0|_{D(M^{(h+1)/2})}^2) =: H_{h,M}, \quad (3.6)$$

where  $L_0$  and  $L_1$  depend only on  $\sigma_M^2$ ,  $K_3$  and  $K_4$ .

**Proposition 3.3** *Let us assume that  $(v_0, v_1) \in D(M^2) \times D(M^{3/2})$ . Then for  $\varepsilon$  small depending only on  $\sigma_M^2$ ,  $K_3$  and  $K_4$  and not on the initial data we have that*

$$e^{2\sigma_M^2 B(t)} \frac{|w_\varepsilon''(t)|^2}{b^4(t)} \leq L_2(\|v_1\|_{D(M^{3/2})}^2 + |v_0|_{D(M^2)}^2), \quad \forall t \geq 0, \quad (3.7)$$

where  $L_2$  depends only on  $\sigma_M^2$ ,  $K_3$  and  $K_4$ .

Now let us prove Proposition 3.2 and Proposition 3.3.

**Proof of Proposition 3.2** Let us denote by  $c_i$  and  $C_i$  various constants that depend only on  $\sigma_M^2$ ,  $K_3$  and  $K_4$ .

The outline of the proof is the following. Firstly (*Step 1*), we prove, for every  $h \geq 0$ , that

if we have that

$$e^{2\sigma_M^2 B(t)} |M^{h/2} v_\varepsilon(t)|^2 \leq R_h \quad \forall t \geq 0 \quad (3.8)$$

then for all  $t \geq 0$  we get that

$$e^{2\sigma_M^2 B(t)} \left[ \varepsilon \frac{|M^{h/2} v'_\varepsilon(t)|^2}{b(t)} + |M^{(h+1)/2} v_\varepsilon(t)|^2 \right] \leq 16\sigma_M^2 R_h + C_0 (\|v_1\|_{D(M^{h/2})}^2 + |v_0|_{D(M^{(h+1)/2})}^2). \quad (3.9)$$

Since problem (3.3) is linear it is enough to prove this estimate for  $h = 0$ .

Then (*Step 2*) we show that for  $h = 1$  we have (3.8) with

$$R_1 = C_1 (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2). \quad (3.10)$$

Using (3.10) in (3.9) with  $h = 1$  we can then conclude that (3.5) holds true if  $h = 1$ . Since (3.3) is linear, (3.5) will be proved for every  $h \geq 1$ .

In conclusion for proving (3.5) we have only to prove (3.9) with  $h = 0$  and (3.10).

Finally (*Step 3*) we prove (3.6). Also in this case it is enough to consider the case  $h = 1$ .

For  $\alpha > 0$  let us introduce the following energies that we use in the proofs:

$$\begin{aligned} D_\alpha(t) &:= e^{2\alpha B(t)} \left[ \langle \varepsilon v'_\varepsilon(t), v_\varepsilon(t) \rangle + \frac{1}{2} |v_\varepsilon(t)|^2 \right], \\ E_\alpha(t) &:= e^{2\alpha B(t)} \left[ \varepsilon \frac{|v'_\varepsilon(t)|^2}{b(t)} + |M^{1/2} v_\varepsilon(t)|^2 \right], \\ F_\alpha(t) &:= e^{2\alpha B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)}. \end{aligned}$$

An easy calculation shows that

$$D'_\alpha = 2\alpha b D_\alpha - b e^{2\alpha B} |M^{1/2} v_\varepsilon|^2 + \varepsilon e^{2\alpha B} |v'_\varepsilon|^2; \quad (3.11)$$

$$E'_\alpha = -e^{2\alpha B} \frac{|v'_\varepsilon|^2}{b} \left( 2 + \varepsilon \frac{b'}{b} - 2\alpha \varepsilon b \right) + 2\alpha b e^{2\alpha B} |M^{1/2} v_\varepsilon|^2; \quad (3.12)$$

$$F'_\alpha = -\frac{1}{\varepsilon} F_\alpha \left( 2 + 2\varepsilon \frac{b'}{b} - 2\alpha \varepsilon b \right) - \frac{2}{\varepsilon} e^{2\alpha B} \frac{1}{b} \langle v'_\varepsilon, M v_\varepsilon \rangle. \quad (3.13)$$

**Step 1 - Proof of (3.9) with  $h = 0$**  Let us choose  $\alpha = 2\sigma_M^2 := \alpha_0$ .

**Estimate on  $D_{\alpha_0}$**  We prove that, if  $\varepsilon$  is small enough, for all  $t \geq 0$  we have that

$$\begin{aligned} \int_0^t e^{2\alpha_0 B(s)} b(s) |M^{1/2} v_\varepsilon(s)|^2 ds &\leq |v_1|^2 + |v_0|^2 + C_2 \varepsilon^2 e^{2\alpha_0 B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} + \\ &+ C_3 \varepsilon \int_0^t e^{2\alpha_0 B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds + 2R_0 e^{\alpha_0 B(t)}. \end{aligned} \quad (3.14)$$

By (3.8) we obtain that

$$\begin{aligned} 2\alpha_0 b D_{\alpha_0} &= 2\alpha_0 \varepsilon b e^{2\alpha_0 B} \langle v'_\varepsilon, v_\varepsilon \rangle + \alpha_0 b e^{2\alpha_0 B} |v_\varepsilon|^2 \\ &\leq \alpha_0 \varepsilon^2 b e^{2\alpha_0 B} |v'_\varepsilon|^2 + 2\alpha_0 b e^{2\alpha_0 B} |v_\varepsilon|^2 \\ &\leq \alpha_0 \varepsilon^2 b^2 e^{2\alpha_0 B} \frac{|v'_\varepsilon|^2}{b} + 2\alpha_0 R_0 b e^{\alpha_0 B}. \end{aligned} \quad (3.15)$$

From (3.11) and (3.15) we therefore get that

$$D'_{\alpha_0} + e^{2\alpha_0 B} b |M^{1/2} v_\varepsilon|^2 \leq \varepsilon (b + \alpha_0 \varepsilon b^2) e^{2\alpha_0 B} \frac{|v'_\varepsilon|^2}{b} + 2\alpha_0 R_0 b e^{\alpha_0 B}. \quad (3.16)$$

Since from (3.2) the function  $b$  is bounded by  $K_4$  then integrating (3.16) we arrive at

$$\int_0^t e^{2\alpha_0 B(s)} b(s) |M^{1/2} v_\varepsilon(s)|^2 ds \leq D_{\alpha_0}(0) - D_{\alpha_0}(t) + \varepsilon c_1 \int_0^t e^{2\alpha_0 B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds + 2R_0 e^{\alpha_0 B(t)}. \quad (3.17)$$

Since  $\varepsilon \leq 1$  and  $b$  is bounded, we can estimate  $D_{\alpha_0}(0)$  and  $D_{\alpha_0}(t)$  as follows:

$$|D_{\alpha_0}(0)| \leq \varepsilon |v_1| |v_0| + \frac{1}{2} |v_0|^2 \leq |v_1|^2 + |v_0|^2, \quad (3.18)$$

$$\begin{aligned} -D_{\alpha_0}(t) &\leq e^{2\alpha_0 B(t)} \left( \varepsilon |v'_\varepsilon(t)| |v_\varepsilon(t)| - \frac{1}{2} |v_\varepsilon(t)|^2 \right) \\ &\leq \frac{1}{2} \varepsilon^2 e^{2\alpha_0 B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} b(t) \leq c_2 \varepsilon^2 e^{2\alpha_0 B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)}. \end{aligned} \quad (3.19)$$

Plugging (3.18) - (3.19) in (3.17) we achieve (3.14).

**Proof of (3.9)** Integrating (3.12) and using (3.14) we get that

$$\begin{aligned} E_{\alpha_0}(t) &\leq E_{\alpha_0}(0) - \int_0^t e^{2\alpha_0 B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} \left( 2 + \varepsilon \frac{b'(s)}{b(s)} - 2\alpha_0 \varepsilon b(s) - 2\alpha_0 C_3 \varepsilon \right) ds + \\ &+ 2\alpha_0 C_2 \varepsilon^2 e^{2\alpha_0 B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} + 2\alpha_0 (|v_1|^2 + |v_0|^2) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}. \end{aligned} \quad (3.20)$$

Thanks to (3.2) we can take  $\varepsilon$  small enough in such a way that

$$2 - 2\varepsilon \sup_{t \geq 0} \frac{|b'(t)|}{b(t)} - 4\sigma_M^2 \varepsilon \sup_{t \geq 0} b(t) - 4\sigma_M^2 \varepsilon C_3 \geq 1, \quad (3.21)$$

$$4\sigma_M^2 C_2 \varepsilon \leq \frac{1}{2}. \quad (3.22)$$

Plugging (3.21) and (3.22) in (3.20) we obtain that

$$E_{\alpha_0}(t) \leq \frac{|v_1|^2}{b(0)} + |M^{1/2}v_0|^2 + \frac{1}{2}E_{\alpha_0}(t) + 2\alpha_0(|v_1|^2 + |v_0|^2) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}$$

from which

$$\frac{1}{2}E_{\alpha_0}(t) \leq c_3(\|v_1\|^2 + |v_0|_{D(M^{1/2})}^2) + 4\alpha_0 R_0 e^{\alpha_0 B(t)}.$$

Since  $\alpha_0 = 2\sigma_M^2$ , hence (3.9) immediately follows dividing all terms by  $e^{\alpha_0 B(t)}$ .

**Step 2 - Proof of (3.10)** For begin with, let us choose

$$\alpha = \sigma_M^2 - \frac{1}{8K_4} := \beta. \quad (3.23)$$

Firstly we prove that for  $\varepsilon$  small and  $h = 0$ ,  $h = 1$  we have for all  $t \geq 0$  that:

$$e^{2\beta B(t)} \left( \varepsilon \frac{|M^{h/2}v'_\varepsilon(t)|^2}{b(t)} + |M^{(h+1)/2}v_\varepsilon(t)|^2 \right) \leq C_4(\|v_1\|_{D(M^{h/2})}^2 + |v_0|_{D(M^{(h+1)/2})}^2), \quad (3.24)$$

and

$$e^{2\beta B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)} \leq C_5(\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2). \quad (3.25)$$

Since (3.3) is linear it is enough to prove (3.24) with  $h = 0$ .

**Estimate on  $D_\beta$**  We prove that, if  $\varepsilon$  is small enough, for all  $t \geq 0$  we have that

$$\begin{aligned} \int_0^t e^{2\beta B(s)} b(s) |M^{1/2}v_\varepsilon(s)|^2 ds &\leq C_6(|v_1|^2 + |v_0|^2) + C_7 \varepsilon^2 e^{2\beta B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} + \\ &+ C_8 \varepsilon \int_0^t e^{2\beta B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds. \end{aligned} \quad (3.26)$$

From (3.1) we obtain that

$$\begin{aligned} 2\beta b D_\beta &= 2\beta \varepsilon b e^{2\beta B} \langle v'_\varepsilon, v_\varepsilon \rangle + \beta b e^{2\beta B} |v_\varepsilon|^2 \\ &\leq \beta \varepsilon b e^{2\beta B} |v'_\varepsilon|^2 + \beta(1 + \varepsilon) b e^{2\beta B} |v_\varepsilon|^2 \\ &\leq \beta \varepsilon b^2 e^{2\beta B} \frac{|v'_\varepsilon|^2}{b} + \frac{\beta}{\sigma_M^2} (1 + \varepsilon) b e^{2\beta B} |M^{1/2}v_\varepsilon|^2. \end{aligned} \quad (3.27)$$

From (3.11) and (3.27) we therefore get that

$$D'_\beta + \left(1 - \frac{\beta}{\sigma_M^2}(1 + \varepsilon)\right) e^{2\beta B} b |M^{1/2} v_\varepsilon|^2 \leq \varepsilon(b + \beta b^2) e^{2\beta B} \frac{|v'_\varepsilon|^2}{b}. \quad (3.28)$$

Since by (3.2) the function  $b$  is bounded then integrating (3.28) we arrive at

$$\begin{aligned} \left(1 - \frac{\beta}{\sigma_M^2}(1 + \varepsilon)\right) \int_0^t e^{2\beta B(s)} b(s) |M^{1/2} v_\varepsilon(s)|^2 ds &\leq D_\beta(0) - D_\beta(t) + \\ &+ \varepsilon c_4 \int_0^t e^{2\beta B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} ds. \end{aligned} \quad (3.29)$$

We can estimate  $D_\beta(0)$  and  $D_\beta(t)$  as in (3.18) and (3.19), furthermore since  $\beta < \sigma_M^2$  we can take  $\varepsilon$  small enough in such a way that

$$1 - \frac{\beta}{\sigma_M^2}(1 + \varepsilon) \geq c_5 > 0. \quad (3.30)$$

Plugging (3.30), (3.18), (3.19) (with  $\beta$  instead of  $\alpha_0$ ) in (3.29) we achieve (3.26).

**Proof of (3.24) with  $h = 0$**  Integrating (3.12) and using (3.26) we get that

$$\begin{aligned} E_\beta(t) &\leq E_\beta(0) - \int_0^t e^{2\beta B(s)} \frac{|v'_\varepsilon(s)|^2}{b(s)} \left(2 + \varepsilon \frac{b'(s)}{b(s)} - 2\beta \varepsilon b(s) - 2\beta \varepsilon C_8\right) ds + \\ &+ 2\beta C_7 \varepsilon^2 e^{2\beta B(t)} \frac{|v'_\varepsilon(t)|^2}{b(t)} + 2\beta C_6(|v_1|^2 + |v_0|^2). \end{aligned} \quad (3.31)$$

Thanks to (3.2) we can take  $\varepsilon$  small enough in such a way that

$$2 - 2\varepsilon \sup_{t \geq 0} \frac{|b'(t)|}{b(t)} - 2\beta \varepsilon \sup_{t \geq 0} b(t) - 2\beta \varepsilon C_8 \geq 1, \quad (3.32)$$

$$2\beta C_7 \varepsilon \leq \frac{1}{2}. \quad (3.33)$$

Plugging (3.32) and (3.33) in (3.31) we obtain that

$$E_\beta(t) \leq \frac{|v_1|^2}{b(0)} + |M^{1/2} v_0|^2 + 2\beta C_6(|v_1|^2 + |v_0|^2) + \frac{1}{2} E_\beta(t),$$

from which (3.24) immediately follows.

**Proof of (3.25)** Plugging (3.32) in (3.13) we have that

$$F'_\beta \leq -\frac{1}{\varepsilon}F_\beta + \frac{2}{\varepsilon}\sqrt{F_\beta}|Mv_\varepsilon|e^{\beta B}.$$

Applying (3.24) with  $h = 1$  we then obtain that

$$F'_\beta \leq -\frac{1}{\varepsilon}\sqrt{F_\beta} \left( \sqrt{F_\beta} - 2\sqrt{C_4(\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2)} \right),$$

hence from Lemma 3.1 we get that

$$F_\beta(t) \leq \max\{F_\beta(0), 4C_4(\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2)\}, \quad \forall t \geq 0,$$

that is (3.25).

**Proof of (3.10)** Since (3.9) holds true, it is enough to prove that (3.8) holds true with  $h = 0$  and

$$R_0 = C_9(\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2). \quad (3.34)$$

To this end let us set  $\alpha = \sigma_M^2$ . By (3.1) we obtain that

$$\begin{aligned} 2\sigma_M^2 b D_{\sigma_M^2} &= 2\sigma_M^2 \varepsilon b e^{2\sigma_M^2 B} \langle v'_\varepsilon, v_\varepsilon \rangle + \sigma_M^2 b e^{2\sigma_M^2 B} |v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B} |v'_\varepsilon|^2 + \sigma_M^2 b^2 e^{2\sigma_M^2 B} |v_\varepsilon|^2 + \sigma_M^2 b e^{2\sigma_M^2 B} |v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B} |v'_\varepsilon|^2 + b^2 e^{2\sigma_M^2 B} |M^{1/2} v_\varepsilon|^2 + b e^{2\sigma_M^2 B} |M^{1/2} v_\varepsilon|^2. \end{aligned} \quad (3.35)$$

Moreover from (3.2) and (3.23) we have that

$$e^{2\sigma_M^2 B(t)} = e^{2\beta B(t)} e^{2(\sigma_M^2 - \beta)B(t)} \leq e^{2\beta B(t)} e^{2K_4(\sigma_M^2 - \beta) \log(1+t)} = e^{2\beta B(t)} (1+t)^{1/4}, \quad (3.36)$$

hence using once again (3.2) and (3.24) with  $h = 0$ , inequality (3.35) becomes

$$\begin{aligned} 2\sigma_M^2 b D_{\sigma_M^2} &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B} |v'_\varepsilon|^2 + c_6(1+t)^{-7/4} e^{2\beta B} |M^{1/2} v_\varepsilon|^2 + b e^{2\sigma_M^2 B} |M^{1/2} v_\varepsilon|^2 \\ &\leq \sigma_M^2 \varepsilon^2 e^{2\sigma_M^2 B} |v'_\varepsilon|^2 + c_7(1+t)^{-7/4} (\|v_1\|^2 + |v_0|_{D(M^{1/2})}^2) + \\ &\quad + b e^{2\sigma_M^2 B} |M^{1/2} v_\varepsilon|^2. \end{aligned} \quad (3.37)$$

Plugging (3.37) into (3.11) and integrating we obtain that

$$D_{\sigma_M^2}(t) \leq D_{\sigma_M^2}(0) + c_8(\|v_1\|^2 + |v_0|_{D(M^{1/2})}^2) + \varepsilon \int_0^t e^{2\sigma_M^2 B(s)} \frac{|v'_\varepsilon(s)|^2}{b^2(s)} b^2(s) (1 + \varepsilon \sigma_M^2) ds. \quad (3.38)$$

From (3.36), (3.25) and (3.2) we get that

$$\begin{aligned}
e^{2\sigma_M^2 B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)} b^2(t) &\leq e^{2\beta B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)} b^2(t) (1+t)^{1/4} \\
&\leq C_5 (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2) b^2(t) (1+t)^{1/4} \\
&\leq c_9 (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2) (1+t)^{-7/4}. \tag{3.39}
\end{aligned}$$

Plugging (3.39) into (3.38) we arrive at

$$\begin{aligned}
\frac{1}{2} e^{2\sigma_M^2 B(t)} |v_\varepsilon(t)|^2 &\leq |D_{\sigma_M^2}(0)| + \varepsilon e^{2\sigma_M^2 B} |\langle v'_\varepsilon(t), v_\varepsilon(t) \rangle| + c_8 (\|v_1\|^2 + |v_0|_{D(M^{1/2})}^2) + \\
&\quad + \varepsilon c_{10} (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2) \\
&\leq c_{11} (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2) + \varepsilon^2 e^{2\sigma_M^2 B(t)} \frac{|v'_\varepsilon(t)|^2}{b^2(t)} b^2(t) + \frac{1}{4} e^{2\sigma_M^2 B(t)} |v_\varepsilon(t)|^2 \\
&\leq c_{12} (\|v_1\|_{D(M^{1/2})}^2 + |v_0|_{D(M)}^2) + \frac{1}{4} e^{2\sigma_M^2 B(t)} |v_\varepsilon(t)|^2.
\end{aligned}$$

By this last inequality (3.34) immediately follows.

**Step 3 - Proof of (3.6) with  $h = 1$**  Let  $\alpha = \sigma_M^2$ . From (3.13), (3.21) and (3.5) used with  $h = 1$  we deduce that

$$F'_{\sigma_M^2} \leq -\frac{1}{\varepsilon} \sqrt{F_{\sigma_M^2}} \left( \sqrt{F_{\sigma_M^2}} - 2|Mv_\varepsilon|e^{\sigma_M^2 B} \right) \leq -\frac{1}{\varepsilon} \sqrt{F_{\sigma_M^2}} \left( \sqrt{F_{\sigma_M^2}} - 2\sqrt{L_{1,M}} \right).$$

We can then apply Lemma 3.1, hence for all  $t \geq 0$  we have that

$$F_{\sigma_M^2}(t) \leq \max\{F_{\sigma_M^2}(0), 4L_{1,M}\} \leq F_{\sigma_M^2}(0) + 4L_{1,M},$$

therefore (3.6) holds true.

□

**Proof of Proposition 3.3** Let us take  $\varepsilon$  small enough in such a way that we can apply Proposition 3.2 (with  $h = 3$  and  $h = 1$ ).

Firstly let us observe that  $w_\varepsilon$  satisfies the following problem

$$\varepsilon w''_\varepsilon(t) + w'_\varepsilon(t) = -b(t)Mv_\varepsilon(t), \quad w_\varepsilon(0) = v_0, \quad w'_\varepsilon(0) = -b(0)Mv_0, \quad w''_\varepsilon(0) = 0. \tag{3.40}$$

If we set

$$G(t) := e^{2\sigma_M^2 B(t)} \frac{|w''_\varepsilon(t)|^2}{b^4(t)},$$



therefore from (3.40) we have that

$$G' = G \left( 2\sigma_M^2 b - 4\frac{b'}{b} \right) - \frac{2}{\varepsilon} \frac{e^{2\sigma_M^2 B}}{b^4} \langle w''_\varepsilon, -w''_\varepsilon - bMv'_\varepsilon - b'Mv_\varepsilon \rangle.$$

Hence we immediately get that

$$G' \leq -\frac{1}{\varepsilon} G \left( 2 - 2\sigma_M^2 b\varepsilon + 4\varepsilon \frac{b'}{b} \right) + \frac{2}{\varepsilon} \sqrt{G} \left( \frac{|Mv'_\varepsilon|}{b} + \frac{|b'|}{b^2} |Mv_\varepsilon| \right) e^{\sigma_M^2 B}. \quad (3.41)$$

Thanks to (3.2) we can take  $\varepsilon$  small enough so that

$$2 - 2\varepsilon\sigma_M^2 \sup_{t \geq 0} b(t) - 4\varepsilon \sup_{t \geq 0} \frac{|b'(t)|}{b(t)} \geq 1. \quad (3.42)$$

Using (3.42), (3.2), (3.5) with  $h = 1$  and (3.6) with  $h = 3$  in (3.41) we obtain that

$$G' \leq -\frac{1}{\varepsilon} \sqrt{G} \left( \sqrt{G} - c_1 (\|v_1\|_{D(M^{3/2})}^2 + |v_0|_{D(M^2)}^2)^{1/2} \right),$$

with a constant  $c_1$  that depends only on  $\sigma_M^2$  and  $K_3, K_4$ . Thus we can apply Lemma 3.1, from which we have that

$$G(t) \leq \max\{G(0), c_1^2 (\|v_1\|_{D(M^{3/2})}^2 + |v_0|_{D(M^2)}^2)\}, \quad \forall t \geq 0.$$

Since  $G(0) = 0$  thesis is proved.  $\square$

### 3.2 Proof of Theorem 2.1

We denote by  $c_{i,\lambda}$  various constants that depend only on  $\lambda$  and on  $|u_0|_{D(A)}, |u_1|_{D(A^{1/2})}$ .

To begin with let us recall that thanks to (2.5) functions  $b_\varepsilon$  verify (3.2) independently of  $\varepsilon$ . Let us also stress that by (2.4) we have  $b_\varepsilon(0) = b_0$  independent of  $\varepsilon$ .

To obtain inequalities (2.11) and (2.12) it is enough to apply Proposition 3.2 with  $M = A_\lambda$ ,  $b(t) = b_\varepsilon(t)$  and  $\sigma_M^2 = \lambda^2$  (taking of course  $\varepsilon$  small enough); indeed in such a case  $U_{\varepsilon,\lambda}$  solves (3.3).

Now let us prove (2.13).

When the initial data are regular we can apply directly Proposition 3.3 with  $M = A_\lambda$  and we obtain (2.13) with a constant that does not depend on  $\varepsilon$ .

Now let us consider the general case in which  $(u_0, u_1) \in D(A) \times D(A^{1/2})$ . Let us set

$$\mu^2 := \lambda^2 + \frac{1}{K_3}$$

where  $K_3$  is the constant in (2.5). Then we can write

$$U_{\varepsilon,\lambda} = V_{\varepsilon,\lambda} + U_{\varepsilon,\mu}, \quad \Theta_{\varepsilon,\lambda} = \theta_{\varepsilon,\lambda} + \Theta_{\varepsilon,\mu}.$$

We estimate separately  $V''_{\varepsilon,\lambda} - \theta''_{\varepsilon,\lambda}$  and  $U''_{\varepsilon,\mu} - \Theta''_{\varepsilon,\mu}$ .

**Estimate on  $U''_{\varepsilon,\mu} - \Theta''_{\varepsilon,\mu}$**  We prove that for every  $t \geq 0$  we have that:

$$e^{2\lambda^2 B_\varepsilon(t)} \frac{1}{b_\varepsilon^4(t)} |U''_{\varepsilon,\mu}(t) - \Theta''_{\varepsilon,\mu}(t)|^2 \leq \frac{c_{1,\lambda}}{\varepsilon^2}. \quad (3.43)$$

Let us assume that  $M = A_\mu$ ,  $b(t) = b_\varepsilon(t)$  and that  $\varepsilon$  is small enough in such a way that we can apply Proposition 3.2 with these choices. Then from (3.3), (3.5), (3.6) with  $h = 1$  and (2.5) we obtain that:

$$\begin{aligned} e^{2\lambda^2 B_\varepsilon(t)} |U''_{\varepsilon,\mu}(t)|^2 &\leq \frac{2}{\varepsilon^2} e^{2(\lambda^2 - \mu^2)B_\varepsilon(t)} e^{2\mu^2 B_\varepsilon(t)} (b_\varepsilon^2(t) |MU_{\varepsilon,\mu}(t)|^2 + |U'_{\varepsilon,\mu}(t)|^2) \\ &\leq \frac{2b_\varepsilon^2(t)}{\varepsilon^2} (L_0 + L_1) (\|u_1\|_{D(A^{1/2})}^2 + |u_0|_{D(A)}^2) e^{2(\lambda^2 - \mu^2)K_3 \log(1+t)} \\ &= \frac{1}{\varepsilon^2} c_{2,\lambda} b_\varepsilon^2(t) (1+t)^{-2}. \end{aligned} \quad (3.44)$$

Moreover  $\Theta_{\varepsilon,\mu}$  verifies (2.10), thence from (2.5) it follows that:

$$\begin{aligned} e^{2\lambda^2 B_\varepsilon(t)} |\Theta''_{\varepsilon,\mu}(t)|^2 &\leq \frac{1}{\varepsilon^2} |\Theta'_{\varepsilon,\mu}(0)|^2 e^{-2t/\varepsilon} e^{2\lambda^2 B_\varepsilon(t)} \\ &\leq \frac{c_{3,\lambda}}{\varepsilon^2} e^{-2t/\varepsilon} e^{2\lambda^2 K_4 \log(1+t)}. \end{aligned} \quad (3.45)$$

Using (3.44), (3.45) and (2.5) we get that

$$\begin{aligned} \frac{e^{2\lambda^2 B_\varepsilon(t)}}{b_\varepsilon^4(t)} |U''_{\varepsilon,\mu}(t) - \Theta''_{\varepsilon,\mu}(t)|^2 &\leq \frac{2}{b_\varepsilon^4(t)} e^{2\lambda^2 B_\varepsilon(t)} (|U''_{\varepsilon,\mu}(t)|^2 + |\Theta''_{\varepsilon,\mu}(t)|^2) \\ &\leq \frac{2c_{2,\lambda}}{\varepsilon^2 b_\varepsilon^2(t)} (1+t)^{-2} + \frac{2c_{3,\lambda}}{\varepsilon^2 b_\varepsilon^4(t)} e^{-2t/\varepsilon} e^{2\lambda^2 K_4 \log(1+t)} \\ &\leq \frac{1}{\varepsilon^2} \left[ c_{4,\lambda} + c_{5,\lambda} (1+t)^4 e^{-t} e^{2\lambda^2 K_4 \log(1+t)} \right] \leq \frac{c_{6,\lambda}}{\varepsilon^2}, \end{aligned}$$

that is (3.43).

**Estimate on  $V''_{\varepsilon,\lambda} - \theta''_{\varepsilon,\lambda}$**  Let  $M = A_{[\lambda,\mu]}$  and  $b(t) = b_\varepsilon(t)$ . Then  $V_{\varepsilon,\lambda}$  and  $\theta_{\varepsilon,\lambda}$  are the solutions of the corresponding problems (3.3) and (3.4). Moreover since  $A_{[\lambda,\mu]}$  is a bounded operator we have that the related initial data  $(v_0, v_1) \in D(M^2) \times D(M^{3/2})$  and

$$|v_0|_{D(M^2)}^2 + |v_1|_{D(M^{3/2})}^2 \leq c_{7,\lambda}.$$

Let  $\varepsilon$  small in such a way that we can apply Proposition 3.3 with these choices. Then since  $w''_\varepsilon = V''_{\varepsilon,\lambda} - \theta''_{\varepsilon,\lambda}$ , and  $\sigma_M^2 = \lambda^2$ , we have that for every  $t \geq 0$ :

$$e^{2\lambda^2 B_\varepsilon(t)} \frac{1}{b_\varepsilon^4(t)} |V''_{\varepsilon,\lambda}(t) - \theta''_{\varepsilon,\lambda}(t)|^2 \leq c_{8,\lambda}. \quad (3.46)$$

**Conclusion** The inequality (2.13) in the general case is a straightforward consequence of (3.43) and (3.46).  $\square$

### 3.3 A decomposition of $u_\varepsilon$

Let  $u_\varepsilon$  be the solution of (1.1) - (1.2) as in Theorem 1 and let  $u_{\varepsilon,\nu}$  be defined as in (2.9). Moreover let us set

$$|A^{1/2}u_\varepsilon(t)|^2 = \nu^2|u_{\varepsilon,\nu}(t)|^2 + \alpha_{\varepsilon,1}(t), \quad |Au_\varepsilon(t)|^2 = \nu^4|u_{\varepsilon,\nu}(t)|^2 + \alpha_{\varepsilon,2}(t), \quad (3.47)$$

$$\frac{|u'_\varepsilon(t)|^2}{b_\varepsilon^2(t)} = \frac{|u'_{\varepsilon,\nu}(t)|^2}{b_\varepsilon^2(t)} + \alpha_{\varepsilon,3}(t), \quad \frac{|A^{1/2}u'_\varepsilon(t)|^2}{b_\varepsilon^2(t)} = \frac{\nu^2|u'_{\varepsilon,\nu}(t)|^2}{b_\varepsilon^2(t)} + \alpha_{\varepsilon,4}(t), \quad (3.48)$$

and

$$e^{2\nu^2 B_\varepsilon(t)}|u_{\varepsilon,\nu}|^2 = \beta_{\varepsilon,0}(t), \quad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,1}(t) = \beta_{\varepsilon,1}(t), \quad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,2}(t) = \beta_{\varepsilon,2}(t), \quad (3.49)$$

$$e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,3}(t) = \beta_{\varepsilon,3}(t), \quad e^{2\nu^2 B_\varepsilon(t)}\alpha_{\varepsilon,4}(t) = \beta_{\varepsilon,4}(t). \quad (3.50)$$

In the proposition below we study the behaviour of quantities defined in (3.49)-(3.50).

**Proposition 3.4** *For  $\varepsilon$  small enough the following properties hold true.*

1. *For  $t \rightarrow +\infty$  we have that:*

$$\beta_{\varepsilon,1}(t) \rightarrow 0, \quad \beta_{\varepsilon,2}(t) \rightarrow 0, \quad \beta_{\varepsilon,3}(t) \rightarrow 0, \quad \beta_{\varepsilon,4}(t) \rightarrow 0, \quad (3.51)$$

$$\beta_{\varepsilon,0}(t) \rightarrow L_\varepsilon \in \mathbb{R} \setminus \{0\}. \quad (3.52)$$

2. *If  $u_{0,\nu} \neq 0$  then there exists a constant  $K_7 > 0$  independent of  $\varepsilon$  and  $t$  such that*

$$\beta_{\varepsilon,0}(t) \geq K_7, \quad \forall t \geq 0. \quad (3.53)$$

**Proof of Proposition 3.4** Let us denote by  $c_i$  various constants that depend only on  $\nu$ ,  $|u_0|_{D(A)}$  and  $|u_1|_{D(A^{1/2})}$ .

**Proof of (3.51)** Let us choose

$$\delta^2 := \nu^2 + \frac{1}{K_3}.$$

Let us assume that  $\varepsilon$  is small enough so that we can use Theorem 2.1 with  $\lambda = \delta$ .

We can rewrite the quantities in (3.47) and (3.48) as:

$$\alpha_{\varepsilon,h}(t) = \sum_{k:\nu < \lambda_k < \delta} \lambda_k^{2h} |u_{\varepsilon,k}(t)|^2 + |A^{h/2} U_{\varepsilon,\delta}(t)|^2 = \alpha_{\varepsilon,h,1}(t) + \alpha_{\varepsilon,h,2}(t), \quad \text{for } h = 1, 2,$$

and for  $h = 3, 4$ :

$$\alpha_{\varepsilon,h}(t) = \frac{1}{b_\varepsilon^2(t)} \left( \sum_{k:\nu < \lambda_k < \delta} \lambda_k^{2(h-3)} |u'_{\varepsilon,k}(t)|^2 + |A^{(h-3)/2} U'_{\varepsilon,\delta}(t)|^2 \right) = \alpha_{\varepsilon,h,1}(t) + \alpha_{\varepsilon,h,2}(t).$$

Since it holds true that:

$$\frac{|A^{h/2} U'_{\varepsilon,\delta}(t)|^2}{b_\varepsilon^2(t)} = \frac{\varepsilon |A^{h/2} U'_{\varepsilon,\delta}(t)|^2}{b_\varepsilon(t)} \frac{1}{\varepsilon} \frac{1}{b_\varepsilon(t)},$$

thence thanks to (2.11) with  $h = 0$  and  $h = 1$  and (2.5) we have that

$$\begin{aligned} e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,2}(t) + \alpha_{\varepsilon,2,2}(t) + \alpha_{\varepsilon,3,2}(t) + \alpha_{\varepsilon,4,2}(t)) &\leq \\ &\leq \frac{c_1}{\varepsilon} \frac{1}{b_\varepsilon(t)} e^{2(\nu^2 - \delta^2) B_\varepsilon(t)} \leq \frac{c_2}{\varepsilon} (1+t) e^{-2 \log(1+t)}. \end{aligned}$$

Hence we get that

$$\lim_{t \rightarrow +\infty} e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,2}(t) + \alpha_{\varepsilon,2,2}(t) + \alpha_{\varepsilon,3,2}(t) + \alpha_{\varepsilon,4,2}(t)) = 0. \quad (3.54)$$

For  $\nu < \lambda_k < \delta$  let us now consider  $b(t) = b_\varepsilon(t)$  and  $M = A_{\{\lambda_k\}}$ . Then from (2.5) the function  $b$  verifies (3.2) and  $M$  verifies (3.1) with  $\sigma_M^2 = \lambda_k^2$ . Let  $\varepsilon$  small enough so that we can apply Proposition 3.2 with such choices. We stress that since  $\nu \leq \lambda_k \leq \delta$  we can take the smallness of  $\varepsilon$  independent of  $\lambda_k$ . Since  $\lambda_k > \nu$  and  $\lambda_k < \delta$  moreover from (3.5) and (3.6) (with  $h = 1$ ) we have that:

$$\begin{aligned} &e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,1}(t) + \alpha_{\varepsilon,2,1}(t) + \alpha_{\varepsilon,3,1}(t) + \alpha_{\varepsilon,4,1}(t)) \\ &\leq c_3 \sum_{k:\nu < \lambda_k < \delta} \left( 2\lambda_k^4 |u_{\varepsilon,k}(t)|^2 + 2 \frac{|u'_{\varepsilon,k}(t)|^2}{b_\varepsilon^2(t)} \right) e^{2\lambda_k^2 B_\varepsilon(t)} e^{2(\nu^2 - \lambda_k^2) B_\varepsilon(t)} \\ &\leq c_4 \sum_{k:\nu < \lambda_k < \delta} (\lambda_k^2 + \lambda_k^4 + 1) (|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2 - \lambda_k^2) B_\varepsilon(t)} \\ &\leq c_5 \sum_{k:\nu < \lambda_k < \delta} (|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2 - \lambda_k^2) B_\varepsilon(t)}. \end{aligned} \quad (3.55)$$

We can therefore passing to the limit in (3.55) so that

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow +\infty} e^{2\nu^2 B_\varepsilon(t)} (\alpha_{\varepsilon,1,1}(t) + \alpha_{\varepsilon,2,1}(t) + \alpha_{\varepsilon,3,1}(t) + \alpha_{\varepsilon,4,1}(t)) \leq \\ &\leq c_5 \sum_{k:\nu < \lambda_k < \delta} \lim_{t \rightarrow +\infty} (|u_{0,k}|^2 + |u_{1,k}|^2) e^{2(\nu^2 - \lambda_k^2) B_\varepsilon(t)} = 0. \end{aligned} \quad (3.56)$$

From (3.54) and (3.56) we get immediately (3.51).

**Proof of (3.52) - (3.53)** Let us set  $y_\varepsilon(t) := |u_{\varepsilon,\nu}(t)|^2$ . Then  $y_\varepsilon$  solves:

$$y'_\varepsilon = -2\nu^2 b_\varepsilon y_\varepsilon - 2\varepsilon \langle u_{\varepsilon,\nu}, u''_{\varepsilon,\nu} \rangle, \quad (3.57)$$

thence for all  $t \geq 0$  we get that

$$e^{2\nu^2 B_\varepsilon(t)} y_\varepsilon(t) = |u_{0,\nu}|^2 - 2\varepsilon \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle e^{2\nu^2 B_\varepsilon(s)} ds. \quad (3.58)$$

Let us now estimate  $\langle u_{\varepsilon,\nu}, u''_{\varepsilon,\nu} \rangle$ . Let us choose  $M = A_{\{\nu\}}$ ,  $b(t) = b_\varepsilon(t)$  and let  $\varepsilon$  small so that we can apply Proposition 3.2 and Proposition 3.3 (with  $v_\varepsilon = u_{\varepsilon,\nu}$ ). This is possible since in such a case

$$|u_{1,\nu}|^2_{D(M^{3/2})} + |u_{0,\nu}|^2_{D(M^2)} \leq c_6(|u_{1,\nu}|^2 + |u_{0,\nu}|^2).$$

Moreover clearly we have that

$$|u_{\varepsilon,\nu}(t)|^2 = \nu^{-4} |Mu_{\varepsilon,\nu}(t)|^2.$$

Then from (3.5) with  $h = 1$  and (3.7), using (2.5) (or equivalently (3.2)) we obtain that

$$\begin{aligned} |\langle u_{\varepsilon,\nu}(t), u''_{\varepsilon,\nu}(t) \rangle| e^{2\nu^2 B_\varepsilon(t)} &\leq |u_{\varepsilon,\nu}(t)| (|w''_\varepsilon(t)| + |\theta''_\varepsilon(t)|) e^{2\nu^2 B_\varepsilon(t)} \\ &\leq c_7(\|u_{1,\nu}\| + |u_{0,\nu}|) \left( \frac{|w''_\varepsilon(t)|}{b_\varepsilon^2(t)} b_\varepsilon^2(t) + \frac{1}{\varepsilon} |\theta'_\varepsilon(0)| e^{-t/\varepsilon} \right) e^{\nu^2 B_\varepsilon(t)} \\ &\leq c_8(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2) \left( b_\varepsilon^2(t) + \frac{1}{\varepsilon} e^{-t/\varepsilon} e^{\nu^2 B_\varepsilon(t)} \right) \\ &\leq c_9(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2) \left( \frac{1}{(1+t)^2} + \frac{1}{\varepsilon} e^{-t/\varepsilon} e^{\nu^2 K_4 \log(1+t)} \right). \end{aligned}$$

Using that

$$\sup_{t \geq 0} e^{-t/2} e^{\nu^2 K_4 \log(1+t)} < +\infty,$$

hence we arrive at

$$|\langle u_{\varepsilon,\nu}(t), u''_{\varepsilon,\nu}(t) \rangle| e^{2\nu^2 B_\varepsilon(t)} \leq c_{10}(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2) \left( \frac{1}{(1+t)^2} + \frac{1}{\varepsilon} e^{-t/2\varepsilon} \right). \quad (3.59)$$

From (3.59) thus we get for all  $t \geq 0$  that

$$\begin{aligned} \left| \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle e^{2\nu^2 B_\varepsilon(s)} ds \right| &\leq \int_0^t |\langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle| e^{2\nu^2 B_\varepsilon(s)} ds \\ &\leq c_{11}(\|u_{1,\nu}\|^2 + |u_{0,\nu}|^2), \end{aligned} \quad (3.60)$$

and also

$$\lim_{t \rightarrow +\infty} \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle e^{2\nu^2 B_\varepsilon(s)} ds = S_\varepsilon \in \mathbb{R}. \quad (3.61)$$

Therefore from (3.58) and (3.61) we have that there exists

$$\lim_{t \rightarrow +\infty} e^{2\nu^2 B_\varepsilon(t)} y_\varepsilon(t) = |u_{0,\nu}|^2 - 2\varepsilon S_\varepsilon.$$

We have to prove that this limit is not zero.

**Case**  $u_{0,\nu} \neq 0$ . By (3.60), for  $\varepsilon$  small we have that

$$2\varepsilon \left| \int_0^t \langle u_{\varepsilon,\nu}(s), u''_{\varepsilon,\nu}(s) \rangle e^{2\nu^2 B_\varepsilon(s)} ds \right| \leq \frac{1}{2} |u_{0,\nu}|^2, \quad \forall t \geq 0,$$

hence (3.53) follows from (3.58) and, as a consequence, the limit in (3.52) is different from zero.

**Case**  $u_{0,\nu} = 0$ . Since  $u_{1,\nu} \neq 0$ , then there exists a single *real* component of  $u_{1,\nu}$  different from zero, that we indicate by  $u_{1,\nu,r}$ . Let  $u_{\varepsilon,\nu,r}$  the related component of  $u_{\varepsilon,\nu}$ . We prove that

$$\lim_{t \rightarrow +\infty} e^{2\nu^2 B_\varepsilon(t)} |u_{\varepsilon,\nu,r}(t)|^2 \neq 0. \quad (3.62)$$

This will be enough to prove that limit in (3.52) is not zero.

To begin with, let us remark that there exists  $T_\varepsilon > 0$  such that

$$u'_{\varepsilon,\nu,r}(T_\varepsilon) = 0.$$

Indeed if it is not the case, then  $u_{\varepsilon,\nu,r}$  is a strictly increasing or decreasing function and since  $u_{\varepsilon,\nu,r}(0) = 0$ , therefore we get that

$$\lim_{t \rightarrow +\infty} u_{\varepsilon,\nu,r}(t) \neq 0,$$

but this is in contrast with (2.1), since  $|A^{1/2}u_\varepsilon(t)|^2 \geq \nu^2 |u_{\varepsilon,\nu,r}(t)|^2$  for all  $t \geq 0$ .

Let now us set

$$T_{\varepsilon,0} := \sup\{\tau \geq 0 : u'_{\varepsilon,\nu,r}(t) \neq 0, \forall t \in [0, \tau]\}.$$

As seen before,  $T_{\varepsilon,0}$  is a real positive number, moreover

$$u'_{\varepsilon,\nu,r}(T_{\varepsilon,0}) = 0,$$

and in  $[0, T_{\varepsilon,0}[$  the function  $u_{\varepsilon,\nu,r}$  is strictly increasing or decreasing, so that

$$u_{\varepsilon,\nu,r}(T_{\varepsilon,0}) = P_\varepsilon \neq 0.$$

Therefore, as in (3.57) - (3.58) for  $t \geq T_{\varepsilon,0}$  we have that

$$e^{2\nu^2(B_\varepsilon(t)-B_\varepsilon(T_{\varepsilon,0}))}|u_{\varepsilon,\nu,r}(t)|^2 = P_\varepsilon^2 - 2\varepsilon \int_{T_{\varepsilon,0}}^t u_{\varepsilon,\nu,r}(s)u''_{\varepsilon,\nu,r}(s)e^{2\nu^2(B_\varepsilon(s)-B_\varepsilon(T_{\varepsilon,0}))}ds. \quad (3.63)$$

Now for  $t \geq 0$ , let us set  $v_\varepsilon(t) = u_{\varepsilon,\nu,r}(t + T_{\varepsilon,0})$ . Then  $v_\varepsilon$  verifies (3.3) with  $M = A_{\{\nu\}}$  restricted to the single component  $u_{\varepsilon,\nu,r}$ ,  $b(t) = b_\varepsilon(t + T_{\varepsilon,0})$  and initial data  $v_\varepsilon(0) = P_\varepsilon$ ,  $v'_\varepsilon(0) = 0$ . Thanks to (2.5) it is clear that the function  $b$  verifies (3.2). Therefore we can obtain as in (3.59) and (3.60):

$$\left| \int_{T_{\varepsilon,0}}^t u_{\varepsilon,\nu,r}(s)u''_{\varepsilon,\nu,r}(s)e^{2\nu^2(B_\varepsilon(s)-B_\varepsilon(T_{\varepsilon,0}))}ds \right| \leq c_{11}P_\varepsilon^2, \quad \forall t \geq T_{\varepsilon,0}. \quad (3.64)$$

Only we have to specify that

$$\sup_{t \geq T_{\varepsilon,0}} e^{-(t-T_{\varepsilon,0})/2} e^{\nu^2 K_4(\log(1+t)-\log(1+T_{\varepsilon,0}))} \leq \sup_{t \geq 0} e^{-t/2} e^{\nu^2 K_4 \log(1+t)} < +\infty.$$

Let now  $\varepsilon$  be small enough so that  $\varepsilon c_{11} \leq 1/2$ , then from (3.63) and (3.64) we get that

$$e^{2\nu^2(B_\varepsilon(t)-B_\varepsilon(T_{\varepsilon,0}))}|u_{\varepsilon,\nu,r}(t)|^2 \geq \frac{1}{2}P_\varepsilon^2, \quad \forall t \geq T_{\varepsilon,0},$$

thus the limit in (3.62) is different from zero.  $\square$

### 3.4 Proof of Theorem 2.3

Let us assume that  $\varepsilon$  is small enough so that Theorem 2.1 with  $\lambda = \nu$  and Proposition 3.4 hold true. Let us moreover denote by  $c_i$  various constants that depend only on  $\nu$ ,  $|u_0|_{D(A)}$  and  $|u_1|_{D(A^{1/2})}$  and by  $c_{i,\varepsilon}$  constants that depend also on  $\varepsilon$ .

**Proof of (2.15)** Since the limit in (3.52) is different from zero, hence there exists  $T_{\varepsilon,1} \geq 0$  such that:

$$\beta_{\varepsilon,0}(t) \geq c_{1,\varepsilon} > 0, \quad \forall t \geq T_{\varepsilon,1}$$

and in particular

$$|u_{\varepsilon,\nu}(t)| > 0, \quad \forall t \geq T_{\varepsilon,1}.$$

Let us remark that if  $u_{0,\nu} \neq 0$ , then thanks to (3.53) we can take  $T_{\varepsilon,1} = 0$ .

Thanks to (3.47) and (2.4) for  $t \geq T_{\varepsilon,1}$  we have that

$$\begin{aligned} b_\varepsilon(t)e^{2\nu^2\gamma B_\varepsilon(t)} &= (\nu^2|u_{\varepsilon,\nu}(t)|^2 + \alpha_{\varepsilon,1}(t))^\gamma e^{2\nu^2\gamma B_\varepsilon(t)} \\ &= \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) \left(1 + \frac{\alpha_{\varepsilon,1}(t)}{\nu^2|u_{\varepsilon,\nu}(t)|^2}\right)^\gamma \\ &= \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) \left(1 + \frac{\beta_{\varepsilon,1}(t)}{\nu^2\beta_{\varepsilon,0}(t)}\right)^\gamma. \end{aligned} \quad (3.65)$$

Since for all  $x \geq 0$  there exists  $0 \leq \xi \leq x$  such that

$$(1+x)^\gamma = 1 + \gamma(1+\xi)^{\gamma-1}x,$$

then for  $t \geq T_{\varepsilon,1}$  we can rewrite (3.65) as

$$b_\varepsilon(t)e^{2\nu^2\gamma B_\varepsilon(t)} = \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) \left(1 + \gamma(1+\xi)^{\gamma-1} \frac{\beta_{\varepsilon,1}(t)}{\nu^2\beta_{\varepsilon,0}(t)}\right) =: \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(t) + \phi_\varepsilon(t), \quad (3.66)$$

where if  $\gamma < 1$  then

$$0 \leq \phi_\varepsilon(t) \leq c_1\beta_{\varepsilon,0}^{\gamma-1}(t)\beta_{\varepsilon,1}(t), \quad (3.67)$$

while if  $\gamma \geq 1$  then

$$0 \leq \phi_\varepsilon(t) \leq c_2\beta_{\varepsilon,0}^\gamma(t) \left(1 + \frac{\beta_{\varepsilon,1}(t)}{\beta_{\varepsilon,0}(t)}\right)^{\gamma-1} \frac{\beta_{\varepsilon,1}(t)}{\beta_{\varepsilon,0}(t)} \leq c_2(\beta_{\varepsilon,0}(t) + \beta_{\varepsilon,1}(t))^{\gamma-1} \beta_{\varepsilon,1}(t). \quad (3.68)$$

Integrating (3.66) we get that

$$e^{2\nu^2\gamma B_\varepsilon(t)} - e^{2\nu^2\gamma B_\varepsilon(T_{\varepsilon,1})} = 2\nu^2\gamma \left[ \int_{T_{\varepsilon,1}}^t \nu^{2\gamma}\beta_{\varepsilon,0}^\gamma(s) + \phi_\varepsilon(s) ds \right], \quad \forall t \geq T_{\varepsilon,1}. \quad (3.69)$$

From (3.52), (3.51) and (3.67) or (3.68) we immediately obtain that

$$\lim_{t \rightarrow +\infty} \frac{1}{1+t} \int_{T_1}^t \beta_{\varepsilon,0}^\gamma(s) ds = L_\varepsilon^\gamma, \quad (3.70)$$

$$\lim_{t \rightarrow +\infty} \phi_\varepsilon(t) = 0, \quad (3.71)$$

$$\lim_{t \rightarrow +\infty} \frac{1}{1+t} \int_{T_1}^t \phi_\varepsilon(s) ds = 0. \quad (3.72)$$

from (3.69), (3.70) and (3.72) we then deduce that

$$\lim_{t \rightarrow +\infty} \frac{1}{1+t} e^{2\nu^2\gamma B_\varepsilon(t)} = 2\nu^{2(\gamma+1)}\gamma L_\varepsilon^\gamma. \quad (3.73)$$

This non zero limit proves (2.15) with constants depending on  $\varepsilon$ .

Let us now assume that  $u_{0,\nu} \neq 0$  so that  $T_{\varepsilon,1} = 0$  and (3.53) holds true. Then from (3.69) we obtain that

$$e^{2\nu^2\gamma B_\varepsilon(t)} \geq 1 + 2\nu^{2(\gamma+1)}\gamma K_7^\gamma t, \quad \forall t \geq 0. \quad (3.74)$$

Moreover since  $u_\varepsilon = U_{\varepsilon,\nu}$ , from (2.11) of Theorem 2.1 (with  $h = 0$  and  $\lambda = \nu$ ) we get that

$$b_\varepsilon(t)e^{2\nu^2\gamma B_\varepsilon(t)} = (e^{2\nu^2 B_\varepsilon(t)} |A^{1/2}u_\varepsilon(t)|^2)^\gamma \leq \gamma_{0,\nu}^\gamma$$

hence for all  $t \geq 0$  we have that

$$e^{2\nu^2\gamma B_\varepsilon(t)} = 1 + 2\nu^2\gamma \int_0^t (e^{2\nu^2 B_\varepsilon(s)} |A^{1/2}u_\varepsilon(s)|^2)^\gamma ds \leq 1 + 2\nu^2\gamma \gamma_{0,\nu}^\gamma t. \quad (3.75)$$

Thus from (3.74) and (3.75) we obtain (2.15) with constants independent of  $\varepsilon$ .



**Proof of (2.16)** From (3.69), for  $t \geq T_{\varepsilon,1}$  we have that

$$B_{\varepsilon}(t) = \frac{1}{2\nu^2\gamma} \log \left( e^{2\nu^2\gamma B_{\varepsilon}(T_{\varepsilon,1})} + 2\nu^2\gamma \int_{T_{\varepsilon,1}}^t \nu^{2\gamma} \beta_{\varepsilon,0}^{\gamma}(s) + \phi_{\varepsilon}(s) ds \right). \quad (3.76)$$

Taking the derivative of (3.76) we obtain that

$$b_{\varepsilon}(t) = \frac{\nu^{2\gamma} \beta_{\varepsilon,0}^{\gamma}(t) + \phi_{\varepsilon}(t)}{e^{2\nu^2\gamma B_{\varepsilon}(T_{\varepsilon,1})} + 2\nu^2\gamma \int_{T_{\varepsilon,1}}^t \nu^{2\gamma} \beta_{\varepsilon,0}^{\gamma}(s) + \phi_{\varepsilon}(s) ds}.$$

Using (3.70), (3.71), (3.72) and (3.52) we get that

$$\lim_{t \rightarrow +\infty} (1+t)b_{\varepsilon}(t) = \lim_{t \rightarrow +\infty} \frac{\nu^{2\gamma} \beta_{\varepsilon,0}^{\gamma}(t) + \phi_{\varepsilon}(t)}{\frac{1}{1+t} \left[ e^{2\nu^2\gamma B_{\varepsilon}(T_{\varepsilon,1})} + 2\nu^2\gamma \int_{T_{\varepsilon,1}}^t \nu^{2\gamma} \beta_{\varepsilon,0}^{\gamma}(s) + \phi_{\varepsilon}(s) ds \right]} = \frac{1}{2\nu^2\gamma},$$

that is (2.16).

**Limit of  $|A^{1/2}u_{\varepsilon}|$**  We prove that

$$\lim_{t \rightarrow +\infty} (1+t)^{1/\gamma} |A^{1/2}u_{\varepsilon}(t)|^2 = \frac{1}{(2\nu^2\gamma)^{1/\gamma}}. \quad (3.77)$$

To this end it is enough to remark that

$$(1+t)^{1/\gamma} |A^{1/2}u_{\varepsilon}(t)|^2 = ((1+t)b_{\varepsilon}(t))^{1/\gamma}$$

and use (2.16).

**Proof of (2.17)** From (3.47) we have that

$$\begin{aligned} (1+t)^{1/\gamma} \nu^2 |u_{\varepsilon,\nu}(t)|^2 &= (1+t)^{1/\gamma} |A^{1/2}u_{\varepsilon}(t)|^2 - (1+t)^{1/\gamma} \alpha_{\varepsilon,1}(t) \\ &= (1+t)^{1/\gamma} |A^{1/2}u_{\varepsilon}(t)|^2 - (1+t)^{1/\gamma} e^{-2\nu^2 B_{\varepsilon}(t)} \beta_{\varepsilon,1}(t). \end{aligned} \quad (3.78)$$

From (2.15) we know that

$$(1+t)^{1/\gamma} e^{-2\nu^2 B_{\varepsilon}(t)} \leq c_{2,\varepsilon}, \quad \forall t \geq 0, \quad (3.79)$$

hence from (3.77), (3.51) and (3.78) we deduce that

$$\lim_{t \rightarrow +\infty} (1+t)^{1/\gamma} \nu^2 |u_{\varepsilon,\nu}(t)|^2 = \frac{1}{(2\nu^2\gamma)^{1/\gamma}}, \quad (3.80)$$

whence (2.17) immediately follows.

**Proof of (2.18)** Thanks to (2.13) in Theorem 2.1 we have for all  $t \geq 0$  that:

$$|U''_{\varepsilon,\nu}(t)| \leq |U''_{\varepsilon,\nu}(t) - \Theta''_{\varepsilon,\nu}(t)| + |\Theta''_{\varepsilon,\nu}(t)| \leq \sqrt{\gamma_{\varepsilon,\nu}} b_{\varepsilon}^2(t) e^{-\nu^2 B_{\varepsilon}(t)} + \frac{1}{\varepsilon} |\Theta'_{\varepsilon,\nu}(0)| e^{-t/\varepsilon}. \quad (3.81)$$

Using (3.79) and (2.5) in (3.81) hence we get that

$$|U''_{\varepsilon,\nu}(t)|^2 \leq c_{3,\varepsilon} \left( \frac{1}{(1+t)^{2+1/(2\gamma)}} + e^{-t} \right)^2 \leq c_{4,\varepsilon} \frac{1}{(1+t)^{4+1/\gamma}}, \quad (3.82)$$

that is (2.18), since  $u_{\varepsilon} = U_{\varepsilon,\nu}$ .

**Existence of  $u_{\varepsilon,\infty}$**  Thanks to (2.8), (3.79), (3.47), (3.49) and (2.5) we have that for all  $t \geq 0$ :

$$(1+t)^{1/\gamma} |u_{\varepsilon}(t) - u_{\varepsilon,\nu}(t)|_{D(A)}^2 \leq c_3 (1+t)^{1/\gamma} |Au_{\varepsilon}(t) - Au_{\varepsilon,\nu}(t)|^2 \leq c_{4,\varepsilon} \beta_{\varepsilon,2}(t),$$

$$(1+t)^{2+1/\gamma} |u'_{\varepsilon}(t) - u'_{\varepsilon,\nu}(t)|_{D(A^{1/2})}^2 \leq c_{5,\varepsilon} (1+t)^2 b_{\varepsilon}^2(t) \beta_{\varepsilon,4}(t) \leq c_{6,\varepsilon} \beta_{\varepsilon,4}(t),$$

hence from (3.51) we obtain that

$$\lim_{t \rightarrow +\infty} (1+t)^{1/\gamma} |u_{\varepsilon}(t) - u_{\varepsilon,\nu}(t)|_{D(A)}^2 + (1+t)^{2+1/\gamma} |u'_{\varepsilon}(t) - u'_{\varepsilon,\nu}(t)|_{D(A^{1/2})}^2 = 0.$$

Therefore for proving (2.14) we have only to show that the functions  $(1+t)^{1/(2\gamma)} u_{\varepsilon,\nu}(t)$  and  $(1+t)^{1+1/(2\gamma)} u'_{\varepsilon,\nu}(t)$  have the required limits. Since

$$u'_{\varepsilon,\nu}(t) = -\nu^2 b_{\varepsilon}(t) u_{\varepsilon,\nu}(t) - \varepsilon u''_{\varepsilon,\nu}(t),$$

then we have that

$$e^{\nu^2 B_{\varepsilon}(t)} u_{\varepsilon,\nu}(t) = u_{0,\nu} - \varepsilon \int_0^t e^{\nu^2 B_{\varepsilon}(s)} u''_{\varepsilon,\nu}(s) ds.$$

Thanks to (2.18) it is clear that for all  $t \geq 0$ :

$$|u''_{\varepsilon,\nu}(t)| \leq |u''_{\varepsilon}(t)| \leq \sqrt{K_{\varepsilon}} \frac{1}{(1+t)^{2+1/(2\gamma)}}, \quad (3.83)$$

thus using once again (2.15) we obtain that there exists

$$\lim_{t \rightarrow +\infty} \int_0^t e^{\nu^2 B_{\varepsilon}(s)} u''_{\varepsilon,\nu}(s) ds = \alpha_{\varepsilon,\nu} \in H_{\{\nu\}}.$$

Applying (3.73) we finally arrive at

$$\begin{aligned}\lim_{t \rightarrow +\infty} (1+t)^{1/(2\gamma)} u_{\varepsilon, \nu}(t) &= \lim_{t \rightarrow +\infty} (1+t)^{1/(2\gamma)} e^{-\nu^2 B_\varepsilon(t)} e^{\nu^2 B_\varepsilon(t)} u_{\varepsilon, \nu}(t) \\ &= \frac{1}{(2\nu^{2(\gamma+1)} \gamma L_\varepsilon^\gamma)^{1/(2\gamma)}} (u_{0, \nu} - \varepsilon \alpha_{\varepsilon, \nu}) = u_{\varepsilon, \infty} \in H_{\{\nu\}}.\end{aligned}\quad (3.84)$$

Furthermore we have also that

$$(1+t)^{1+1/(2\gamma)} u'_{\varepsilon, \nu}(t) = -\nu^2 (1+t) b_\varepsilon(t) (1+t)^{1/(2\gamma)} u_{\varepsilon, \nu}(t) - \varepsilon (1+t)^{1+1/(2\gamma)} u''_{\varepsilon, \nu}(t),$$

therefore from (2.16), (3.84) and (3.83) we get that

$$\lim_{t \rightarrow +\infty} (1+t)^{1+1/(2\gamma)} u'_{\varepsilon, \nu}(t) = -\frac{1}{2\gamma} u_{\varepsilon, \infty} \in H_{\{\nu\}}. \quad (3.85)$$

From (3.84), (3.85) and (2.17) the existence of the required non zero limits immediately follows.  $\square$

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